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Abstract In this paper, we prove a Meyers' type estimate for weak solutions to a Stokes system with bounded measurable coefficients in place of the usual constant viscosity. Besides the perturbation argument due to Meyers, we make use of the solvability of the classical Stokes problem in $[W^{1,q}_{0,\sigma}(\Omega)]^n$ ($n = 2$ or $n = 3, 2 < q < 3 + \varepsilon, \partial\Omega$ Lipschitz).

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ ($n = 2$ or $n = 3$) be a bounded domain with Lipschitz boundary $\partial\Omega$. We consider the following generalized Navier-Stokes problem:

$$-\frac{\partial}{\partial x_j}(A_{ij}^{kl} D_{kl}(\boldsymbol{u})) + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} = -\frac{\partial f_{ij}}{\partial x_j} \quad \text{in} \quad \Omega \quad (i = 1, \dots, n)\,^{1)}, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega, \tag{1.2}$$

$$\boldsymbol{u} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.3}$$

where

$$\begin{aligned} \boldsymbol{u} &= (u_1, \dots, u_n) = \text{ velocity}, \; p = \text{ pressure}, \\ D(\boldsymbol{u}) &= \{D_{ij}(\boldsymbol{u})\} = \text{ rate of strain}, \; D_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \\ \boldsymbol{f} &= \{f_{ij}\} = \text{ external force.} \end{aligned}$$

The coefficients $A_{ij}^{kl} = A_{ij}^{kl}(x)$ are assumed to satisfy the following conditions:

$$A_{ij}^{kl} \quad \text{is measurable in} \quad \Omega \quad (i, j, k, l = 1, \dots, n) \tag{1.4$_1$}$$

$$\begin{cases} A_{ij}^{kl}(x) = A_{kl}^{ij}(x), \\ A_{ij}^{kl}(x)\xi_{kl} = A_{ji}^{kl}(x)\xi_{kl} \quad \text{for all symmetric } \boldsymbol{\xi} = \{\xi_{ij}\} \in \mathbb{R}^{n^2}, \\ \text{for a.e.} \quad x \in \Omega \ (i, j, k, l = 1, \dots, n); \end{cases} \tag{1.4$_2$}$$

$$\begin{cases} \exists\, \alpha_1, \alpha_2 > 0 : \ \alpha_1 |\boldsymbol{\xi}|^2 \le A_{ij}^{kl}(x)\xi_{kl}\xi_{ij} \le \alpha_2 |\boldsymbol{\xi}|^2 \\ \text{for all} \quad \boldsymbol{\xi} = \{\xi_{ij}\} \in \mathbb{R}^{n^2}, \ \text{for a.e.} \quad x \in \Omega. \end{cases} \tag{1.4$_3$}$$

In many cases of practical interest, the viscosity ν of a fluid may significantly depend on functions such as the temperature θ of the fluid, or the turbulent energy k of the fluid motion, i. e. $\nu = \nu(\theta(x))$ or $\nu = \nu(k(x))$, respectively ($x \in \Omega$). Then system (1.1), (1.2) has to be completed by an equation which must be satisfied by θ or k, respectively. Both types of viscosities are included in the following special case of coefficients:

$$A_{ij}^{kl}(x) = a(x)\delta_{ik}\delta_{jl}, \quad x \in \Omega,$$

where

$$\begin{cases} a \quad \textit{is measurable in} \quad \Omega, \\ \exists\, \alpha_1, \alpha_2 > 0 : \alpha_1 \le a(x) \le \alpha_2 \quad \textit{for a. e.} \quad x \in \Omega. \end{cases}$$

1) Throughout repeated indices imply summation on $1, \dots, n$. For $\boldsymbol{\xi} = \{\xi_{ij}\} \in \mathbb{R}^{n^2}$, set $|\boldsymbol{\xi}|^2 = \xi_{ij}\xi_{ij}$.

■

It is readily seen that these coefficients $A^{kl}_{ij}(x)$ satisfy condition (1.4).

In particular, if

$$A^{kl}_{ij}(x) = a_0\delta_{ik}\delta_{jl} \quad \forall\, x \in \Omega,\ a_0 = \text{const} > 0,$$

then (1.1), (1.2) turns into the classical stationary Navier-Stokes system with viscosity $\nu = \dfrac{a_0}{2}$ $\left(\text{note that } \dfrac{\partial}{\partial x_j} D_{ij}(\boldsymbol{v}) = \dfrac{1}{2}\Delta v_i \ (i = 1, \ldots, n) \text{ whenever } \operatorname{div} \boldsymbol{v} = 0\right)$.

■

To define the concept of weak solution to (1.1)-(1.3) we introduce the following notations. By $W^{1,q}(\Omega)$ $(1 \le q \le +\infty)$ we denote the usual Sobolev space. Set

$$\begin{aligned} W^{1,q}_0(\Omega) &:= \{\varphi \in W^{1,q}(\Omega); \varphi = 0 \text{ a.e. on } \partial\Omega\}, \\ [W^{1,q}_{0,\sigma}(\Omega)]^n &:= \{\boldsymbol{v} \in [W^{1,q}_0(\Omega)]^n; \operatorname{div} \boldsymbol{v} = 0 \text{ a.e. in } \Omega\}. \end{aligned}$$

By Korn's inequality, $[W^{1,q}_{0,\sigma}(\Omega)]^n$ is a Banach space with respect to the norm

$$\|\boldsymbol{v}\|_{[W^{1,q}_{0,\sigma}(\Omega)]^n} = \left(\int\limits_\Omega |D(\boldsymbol{v})|^q \right)^{\frac{1}{q}}.$$

If $q = 2$, it is easy to check that

$$\int\limits_\Omega |D(\boldsymbol{v})|^2 = \frac{1}{2}\int\limits_\Omega |\nabla \boldsymbol{v}|^2 \quad \forall\, \boldsymbol{v} \in [W^{1,2}_{0,\sigma}(\Omega)]^n. \tag{1.5}$$

Next, for $\boldsymbol{u}, \boldsymbol{v} \in [W^{1,q}(\Omega)]^n$ define

$$\boldsymbol{A}(\boldsymbol{u}, \boldsymbol{v}) := A^{kl}_{ij} D_{kl}(\boldsymbol{u}) D_{ij}(\boldsymbol{v}) \quad \text{a.e. in } \Omega.$$

Then from (1.4) and (1.5) it follows that $[W^{1,2}_{0,\sigma}(\Omega)]^n$ is a Hilbert space with respect to the scalar product

$$((\boldsymbol{u}, \boldsymbol{v})) := \int\limits_\Omega \boldsymbol{A}(\boldsymbol{u}, \boldsymbol{v}).$$

■

Assume (1.4). Let $\boldsymbol{f} \in [L^2(\Omega)]^{n^2}$. *The vector field* $\boldsymbol{u} \in [W^{1,2}_{0,\sigma}(\Omega)]^n$ *is called* **weak solution to (1.1)-(1.3)** *if*

$$\int\limits_\Omega \boldsymbol{A}(\boldsymbol{u}, \boldsymbol{v}) + \int\limits_\Omega u_i \frac{\partial u_j}{\partial x_i} v_j = \int\limits_\Omega \boldsymbol{f} : \nabla \boldsymbol{v} \ ^{2)} \quad \forall\, \boldsymbol{v} \in [W^{1,2}_{0,\sigma}(\Omega)]^n. \tag{1.6}$$

[2)] For $\boldsymbol{\xi} = \{\xi_{ij}\}$, $\boldsymbol{\eta} = \{\eta_{ij}\} \in \mathbb{R}^{n^2}$ define $\boldsymbol{\xi} : \boldsymbol{\eta} = \xi_{ij}\eta_{ij}$.

The existence of a weak solution to (1.1)-(1.3) can be proved by the same arguments as for the case of (classical) Navier-Stokes equations. Once a weak solution $\boldsymbol{u} \in [W^{1,2}_{0,\sigma}(\Omega)]^n$ to (1.1)-(1.3) is at hand, we obtain a $\hat{p} \in L^2(\Omega)/\mathbb{R}$ such that

$$\int\limits_\Omega \boldsymbol{A}(\boldsymbol{u}, \boldsymbol{\varphi}) + \int\limits_\Omega u_i \frac{\partial u_j}{\partial x_i} \varphi_j = \int\limits_\Omega p \operatorname{div} \boldsymbol{\varphi} + \int\limits_\Omega \boldsymbol{f} : \nabla \boldsymbol{\varphi}$$

for all $\boldsymbol{\varphi} \in [W^{1,2}_0(\Omega)]^n$ and all representatives $p \in \hat{p}$ (see, e. g., Sohr [12; Lemma 2.2.2, p. 75]).

■

Remark Without loss of generality we may restrict our discussion of (1.1)-(1.3) to the right hand sides of the form $-\dfrac{\partial f_{ij}}{\partial x_j}$ considered in (1.1). Indeed, the more general right hand sides

$$(*) \qquad f_{0i} - \frac{\partial f_{ij}}{\partial x_j} \qquad (i = 1, \ldots, n)$$

can be included in our discussion as follows. Let $\boldsymbol{f}_0 \in [L^r(\Omega)]^n \quad (1 < r < 2$ if $n = 2$, $r = \frac{6}{5}$). Set

$$a_i = \frac{1}{\operatorname{mes} \Omega} \int\limits_\Omega f_{0i} \qquad (i = 1, \ldots, n).$$

From Sohr [12; Lemma 2.1.1] we obtain the existence of a $\boldsymbol{g}_i \in [W^{1,r}_0(\Omega)]^n$ such that

$$\operatorname{div} \boldsymbol{g}_i = f_{0i} - a_i \text{ a. e. in } \quad \Omega, \quad \|\nabla \boldsymbol{g}_i\|_{L^r} \le c \|f_{0i}\|_{L^r}.$$

Define

$$h_{ij}(x) := g_{ij}(x) + a_i x_j \quad \text{for a. e. } x \in \Omega, \quad i, j = 1, \ldots, n.$$

Then

$$h_{ij} \in L^{r^*}(\Omega) \quad \Big(r^* := \frac{2r}{2 - r} \text{ if } n = 2, \quad r^* := 2 \text{ if } n = 3 \Big),$$

$$\int\limits_\Omega f_{0i} \varphi_i = - \int\limits_\Omega h_{ij} \frac{\partial \varphi_i}{\partial x_j} \quad \forall \boldsymbol{\varphi} \in [W^{1,2}_0(\Omega)]^n.$$

In this sense, $(*)$ can be replaced by

$$- \frac{\partial}{\partial x_j} (h_{ij} + f_{ij}).$$

Finally, given $q > 2$, define $r := \dfrac{nq}{n + q}$. Clearly, $1 < r < 2$ if $n = 2$, $\dfrac{6}{5} < r < 3$ if $n = 3$, and $q = r^* = \dfrac{nr}{n - r}$. Then Theorem 1 and 2 apply to (1.1)-(1.3) with right hand sides of the form $(*)$ ■

The aim of the present paper is to prove the following higher integrability result: there exists an $r > 2$ such that

$$D(\boldsymbol{u}) \in [L^r(\Omega)]^{n^2}$$

for every weak solution $\boldsymbol{u} \in [W^{1,2}_{0,\sigma}(\Omega)]^n$ to (1.1)-(1.3).

2. Statement of the main result

Theorem 1 *Assume*

$$\begin{aligned} &\partial\Omega \quad \textit{is Lipschitz},\\ 2 < q < 3 + \varepsilon \quad &(\varepsilon > 0 \textit{ according to Brown; Shen [2]}). \end{aligned}$$

Let $\boldsymbol{f} = \{f_{ij}\} \in [L^q(\Omega)]^{n^2}$. *Define*

$$s = \begin{cases} q & \textit{if} \quad n = 2,\\ \min\{q, 3\} & \textit{if} \quad n = 3. \end{cases}$$

Then, there exists an $r \in]2, s[$ *and a constant* $C_0 = C_0(r, s) > 0$ *satisfying*

$$C_0\Big(1 - \frac{\alpha_1}{\alpha_2}\Big) < 1$$

such that for every weak solution $\boldsymbol{u} \in [W^{1,2}_{0,\sigma}(\Omega)]^n$ *to* (1.1)-(1.3) *there holds*

$$\begin{aligned} D(\boldsymbol{u}) &\in [L^r(\Omega)]^{n^2},\\ \bigg(1 - C_0\Big(1 - \frac{\alpha_1}{\alpha_2}\Big)\bigg)\|D(\boldsymbol{u})\|_{L^r} &\le C_1(\|\boldsymbol{f}\|_{L^r} + \|\boldsymbol{f}\|^2_{L^2}), \end{aligned}$$

where $C_1 = \mathrm{const} > 0$ *depends on* $r, s, \operatorname{mes}\Omega$ *and the constant of embedding* $W^{1,2}_0(\Omega) \subset L^{2^*}(\Omega)$ [3] .

The proof of this result relies on the following two cornerstones. Firstly, a theorem on the existence and uniqueness of a weak solution in $[W^{1,q}_{0,\sigma}(\Omega)]^n$ to the Stokes problem (see Proposition 1 below). Secondly, a perturbation argument due to Meyers [9] for proving the higher integrability of the gradient of weak solutions to linear elliptic equations. This idea has been developed further for large classes of linear elliptic systems in divergence form by Nečas [10], [11] and Stará [13] (see also Giaquinta [6; pp. 154-157]).

The technique of reverse Hölder inequality for proving the higher integrability of the gradient of weak solutions to nonlinear elliptic systems has been applied by Druet [4], and by Wolff [14] to the stationary Navier-Stokes system, and by Zhikov [15] to a nonlinear stationary Stokes system. ■

[3] 2^* = Sobolev embedding exponent.

3. Preliminaries

Let $\Omega \subset \mathbb{R}^n \quad (n = 2 \text{ or } n = 3)$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We consider the Stokes problem

$$
\begin{aligned}
&(3.1) \qquad & -\Delta \boldsymbol{w} + \nabla p &= -\operatorname{div} \boldsymbol{F} \quad && \text{in} \quad \Omega \\
&(3.2) \qquad & \operatorname{div} \boldsymbol{w} &= 0 && \text{in} \quad \Omega, \\
&(3.3) \qquad & \boldsymbol{w} &= 0 && \text{on} \quad \partial\Omega.
\end{aligned}
$$

$\boxed{3.1}$ The space $[W^{1,2}_{0,\sigma}(\Omega)]^n$ is a Hilbert with respect to the scalar product

$$(\boldsymbol{u}, \boldsymbol{v})_{W^{1,2}_{0,\sigma}} = \int\limits_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v}.$$

Let $\boldsymbol{F} \in [L^2(\Omega)]^{n^2}$. An application of the Riesz representation theorem to the linear continuous functional $\boldsymbol{v} \mapsto \int\limits_\Omega \boldsymbol{F} : \nabla \boldsymbol{v}$ provides the existence and uniqueness of a $\boldsymbol{w} \in [W^{1,2}_{0,\sigma}(\Omega)]^n$ such that

$$(3.4_n) \qquad \left\{ \begin{aligned} \int\limits_\Omega \nabla \boldsymbol{w} : \nabla \boldsymbol{v} &= \int\limits_\Omega \boldsymbol{F} : \nabla \boldsymbol{v} \quad \forall\, \boldsymbol{v} \in [W^{1,2}_{0,\sigma}(\Omega)]^n, \\ \|\nabla \boldsymbol{w}\|_{L^2} &\le \|\boldsymbol{F}\|_{L^2}. \end{aligned} \right.$$

$\boxed{3.2}$ **Proposition 1** *Let*

$$2 < q < 3 + \varepsilon \quad (\varepsilon > 0 \textit{ according to Brown; Shen [2]}).$$

Then, for every $\boldsymbol{F} \in [L^q(\Omega)]^{n^2}$ *there exists exactly one* $\boldsymbol{w} \in [W^{1,q}_{0,\sigma}(\Omega)]^n$ *such that*

$$
\begin{aligned}
\int\limits_\Omega \nabla \boldsymbol{w} : \nabla \boldsymbol{v} &= \int\limits_\Omega \boldsymbol{F} : \nabla \boldsymbol{v} \quad \forall\, \boldsymbol{v} \in [W^{1,q'}_{0,\sigma}(\Omega)]^n \quad \Big(q' = \frac{q}{q-1}\Big), \\
\|\nabla \boldsymbol{w}\|_{L^2} &\le c(q) \|\boldsymbol{F}\|_{L^2}.
\end{aligned}
$$

Proof If $n = 3$, the assertion follows from Brown; Shen [2] [4].

Consider $n = 2$. There exists exactly one $\boldsymbol{w} \in [W^{1,2}_{0,\sigma}(\Omega)]^2$ such that

$$(3.4_2) \qquad \left\{ \begin{aligned} \int\limits_\Omega \nabla \boldsymbol{w} : \nabla \boldsymbol{v} &= \int\limits_\Omega \boldsymbol{F} : \nabla \boldsymbol{v} \quad \forall\, \boldsymbol{v} \in [W^{1,2}_{0,\sigma}(\Omega)]^2, \\ \|\nabla \boldsymbol{w}\|_{L^2} &\le \|\boldsymbol{F}\|_{L^2}. \end{aligned} \right.$$

[4] We note that the result to which we refer, is in fact true for $\frac{3+\varepsilon}{2+\varepsilon} < q < 3 + \varepsilon$ (see [2; p. 1191]). However, for our purposes it is enough to consider $2 < q < 3 + \varepsilon$.

Again using Sohr [12] (loc. cit.) we obtain the existence and uniqueness of a $p \in L^2(\Omega)$ such that

$$
(3.5_2) \qquad \begin{cases} \displaystyle\int\limits_{\Omega} p \, = \, 0, \\ \displaystyle\int\limits_{\Omega} \nabla \boldsymbol{\varphi} : \nabla \boldsymbol{v} \, = \, \int\limits_{\Omega} p \operatorname{div} \boldsymbol{\varphi} + \int\limits_{\Omega} \boldsymbol{F} : \nabla \boldsymbol{\varphi} \quad \forall\, \boldsymbol{\varphi} \in [W_0^{1,2}(\Omega)]^2, \\ \|p\|_{L^2} \, \le \, c \|\boldsymbol{F}\|_{L^2}. \end{cases}
$$

We prove: $\boldsymbol{w} \in [W^{1,q}_{0,\sigma}(\Omega)]^2$, $\|\nabla \boldsymbol{w}\|_{L^q} \, \le \, c(q)\|\boldsymbol{F}\|_{L^q}$.

To this end, fix $\zeta \in C^\infty_c(]0,1[)$ such that $0 \le \zeta \le 1$ in $]0,1[$ and $\zeta = 1$ in $\left[\frac{1}{3}, \frac{2}{3}\right]$. Define $\widetilde{\Omega} := \Omega \times]0,1[$. It is easy to check that $\widetilde{\Omega}$ is a bounded Lipschitz domain in $\mathbb{R}^3$. For a. e. $x = (x_1, x_2, x_3) \in \widetilde{\Omega}$, define

$$
z_i(x) := w_i(x_1, x_2)\zeta(x_3) \quad (i = 1, 2), \quad z_3(x) = 0.
$$

It follows that, for a. e. $x \in \widetilde{\Omega}$

$$
\begin{aligned}
&\frac{\partial z_i}{\partial x_j} \, = \, \frac{\partial w_i}{\partial x_j}\zeta, \quad \frac{\partial z_i}{\partial x_3} \, = \, w_i \zeta' \quad (i, j = 1, 2), \\
&\frac{\partial z_3}{\partial x_k} \, = \, 0 \quad (k = 1, 2, 3), \\
&\operatorname{div} \boldsymbol{z} \, = \, 0
\end{aligned}
$$

and $\boldsymbol{z} = 0$ a. e. on $\partial\widetilde{\Omega}$. Thus, $\boldsymbol{z} \in [W^{1,2}_{0,\sigma}(\widetilde{\Omega})]^3$.

We show that $\boldsymbol{z}$ is the weak solution to a Stokes problem in $\widetilde{\Omega}$. Indeed, let $\boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3) \in [W^{1,2}_{0,\sigma}(\widetilde{\Omega})]^3$. We obtain

$$
\begin{aligned}
\int\limits_{\widetilde{\Omega}} \nabla \boldsymbol{z} : \nabla \boldsymbol{\psi} dx \, &= \, \int\limits_{\widetilde{\Omega}} \Big(\sum_{i,j=1}^{2} \frac{\partial z_i}{\partial x_j} \frac{\partial \psi_i}{\partial x_j} + \sum_{i=1}^{2} \frac{\partial z_i}{\partial x_3} \frac{\partial \psi_i}{\partial x_3} \Big) dx \\
&= \int\limits_0^1 \Bigg(\sum_{i,j=1}^{2} \int\limits_{\Omega} \frac{\partial w_i}{\partial x_j} \frac{\partial \psi_i}{\partial x_j} dx_1 dx_2 \Bigg) \zeta dx_3 + \sum_{i=1}^{2} \int\limits_{\widetilde{\Omega}} w_i \zeta' \frac{\partial \psi_i}{\partial x_3} dx \\
&\quad \text{[by } (3.5_2)] \\
(3.6_3) \qquad &= \int\limits_0^1 \Bigg(\int\limits_{\Omega} p \Big(\frac{\partial \psi_1}{\partial x_1} + \frac{\partial \psi_2}{\partial x_2} \Big) dx_1 dx_2 + \sum_{i,j=1}^{2} \int\limits_{\Omega} F_{ij} \frac{\partial \psi_i}{\partial x_j} dx_1 dx_2 \Bigg) \zeta dx_3 \\
&\quad + \sum_{i=1}^{2} \int\limits_{\widetilde{\Omega}} w_i \zeta' \frac{\partial \psi_i}{\partial x_3} dx.
\end{aligned}
$$

Observing that $\dfrac{\partial\psi_1}{\partial x_1} + \dfrac{\partial\psi_2}{\partial x_2} = -\dfrac{\partial\psi_3}{\partial x_3}$ a. e. in Ω, we find

$$\int\limits_0^1 \Big(\int\limits_\Omega p\Big(\frac{\partial\psi_1}{\partial x_1} + \frac{\partial\psi_2}{\partial x_2}\Big) dx_1 dx_2\Big)\zeta dx_3 = \int\limits_{\widetilde\Omega} p\,\zeta'\psi_3 dx.$$

Then (3.6_3) takes the form

$$(3.7_3) \qquad \int\limits_{\widetilde\Omega} \nabla\boldsymbol{z} : \nabla\boldsymbol{\psi} = \int\limits_{\widetilde\Omega} \Big(p\,\zeta'\psi_3 + \sum_{i,j=1}^{2} F_{ij}\zeta\frac{\partial\psi_i}{\partial x_j} + \sum_{i=1}^{2} w_i\zeta'\frac{\partial\psi_i}{\partial x_3}\Big).$$

By Hölder's inequality and Sobolev's embedding theorem,

$$\begin{aligned}
&\Big|\int\limits_{\widetilde\Omega} \Big(p\,\zeta'\psi_3 + \sum_{i,j=1}^{2} F_{ij}\zeta\frac{\partial\psi_i}{\partial x_j} + \sum_{i=1}^{2} w_i\zeta'\frac{\partial\psi_i}{\partial x_3}\Big)\Big| \le \\
&\quad \le \max|\zeta'|\,\|p\|_{L^2(\Omega)}\|\psi_3\|_{L^2(\widetilde\Omega)} + \sum_{i,j=1}^{2}\|F_{ij}\|_{L^q(\Omega)}\Big\|\frac{\partial\psi_i}{\partial x_3}\Big\|_{L^{q'}(\widetilde\Omega)} \\
&\qquad + \max|\zeta'|\sum_{i=1}^{2}\|w_i\|_{L^q(\Omega)}\Big\|\frac{\partial\psi_i}{\partial x_3}\Big\|_{L^{q'}(\widetilde\Omega)} \\
&\quad \le c\|\boldsymbol{F}\|_{[L^q(\Omega)]^4}\|\boldsymbol{\psi}\|_{[W^{1,q'}(\widetilde\Omega)]^3} \qquad \text{[by } (3.4_2),\ (3.5_2)]
\end{aligned}$$

for all $\boldsymbol{\psi} \in [W_0^{1,q'}(\widetilde\Omega)]^3$ $(2 < q < 3+\varepsilon$; without loss of generality, we may assume $\varepsilon < 3)$.

In other words, $\boldsymbol{z} \in [W_{0,\sigma}^{1,2}(\widetilde\Omega)]^3$ is the weak solution to the Stokes problem

$$(3.8) \qquad -\Delta\boldsymbol{z} + \nabla\tilde p = -\operatorname{div}\widetilde{\boldsymbol{F}} \quad \text{in} \quad \widetilde\Omega,$$

$$(3.9) \qquad \operatorname{div}\boldsymbol{z} = 0 \quad \text{in} \quad \widetilde\Omega,$$

$$(3.10) \qquad \boldsymbol{z} = 0 \quad \text{on} \quad \partial\widetilde\Omega,$$

where

$$\widetilde{\boldsymbol{F}} \in [W^{-1,q}(\widetilde\Omega)]^9, \quad \|\widetilde{\boldsymbol{F}}\|_{[W^{-1,q}(\widetilde\Omega)]^9} \le c\|\boldsymbol{F}\|_{[L^q(\Omega)]^4}.$$

Applying Brown; Shen [2] to (3.8)-(3.10) we obtain the existence and uniqueness of a $\boldsymbol{z}^* \in [W_{0,\sigma}^{1,q}(\widetilde\Omega)]^3$ such that

$$\int\limits_{\widetilde\Omega} \nabla\boldsymbol{z}^* : \nabla\boldsymbol{\psi} = \int\limits_{\widetilde\Omega} \Big(p\,\zeta'\psi_3 + \sum_{i,j=1}^{2} F_{ij}\zeta\frac{\partial\psi_i}{\partial x_j} + \sum_{i=1}^{2} w_i\zeta'\frac{\partial\psi_i}{\partial x_3}\Big)$$

for all $\boldsymbol{\psi} \in [W_{0,\sigma}^{1,q'}(\widetilde\Omega)]^3$, and

$$\|\nabla\boldsymbol{z}^*\|_{W^{1,q}} \le c(q)\|\widetilde{\boldsymbol{F}}\|_{W^{-1,q}}.$$

By (3.7_3),

$$\int\limits_{\widetilde{\Omega}} \nabla(\boldsymbol{z} - \boldsymbol{z}^*) : \nabla \boldsymbol{\psi} \; = \; 0 \quad \forall\, \boldsymbol{\psi} \in [W^{1,\,2}_{0,\sigma}(\widetilde{\Omega})]^3.$$

Therefore $\boldsymbol{z} = \boldsymbol{z}^*$ a. e. in $\widetilde{\Omega}$. Recalling that

$$\frac{\partial w_i}{\partial x_j}\zeta \; = \; \frac{\partial z_i}{\partial x_j} \quad \text{a. e. in} \quad \widetilde{\Omega} \quad (i,j = 1,2)$$

we obtain

$$\int\limits_{\Omega} \Big|\frac{\partial w_i}{\partial x_j}\Big|^q dx_1 dx_2 \; \leq \; 3 \int\limits_{\widetilde{\Omega}} \Big|\frac{\partial z_i}{\partial x_j}\Big|^q dx \quad (i,j = 1,2),$$

and finally

$$\|\nabla \boldsymbol{w}\|_{L^q} \; \leq \; c_2(q)\|\boldsymbol{F}\|_{L^q}.$$

■

Remarks 1. Galdi; Simader; Sohr [5] proved the existence and uniqueness of a weak solution $\boldsymbol{z} \in [W^{1,\,q}_{0,\sigma}(\Omega)]^n$ to the Stokes problem (3.8)-(3.10) for any dimension $n \geq 2$, for every $1 < q < +\infty$ and for bounded domains $\Omega \subset \mathbb{R}^n$ whose boundary $\partial\Omega$ is Lipschitz with sufficiently small Lipschitz constant L. Here "smallness of L" means: L is smaller than a certain constant depending only on n and q.

2. Regularity results for (3.8)-(3.10) have been proved by Kellog; Osborn [8] when $n = 2$ and Ω is a convex polygon [note that every convex domain $\Omega \subset \mathbb{R}^n$ has Lipschitz continuous boundary $\partial\Omega$].

3. Amrouche; Girault [1] solved (3.8)-(3.10) in $[W^{1,\,q}_{0,\sigma}(\Omega)]^n$ $(1 < q < +\infty)$ for bounded domains Ω with boundary of class $\mathcal{C}^{1,1}$.

From Prop. 1 we deduce

Proposition 1′ *Let*

$$2 \leq q < 3 + \varepsilon \quad (\varepsilon > 0 \textit{ according to Brown; Shen [2]}).$$

Then, for every $\boldsymbol{F} \in [L^q_{\mathrm{sym}}(\Omega)]^{n^2}$ [5] *there exists exactly one* $\boldsymbol{w} \in [W^{1,\,q}_{0,\sigma}(\Omega)]^n$ *such that*

$$(3.11) \qquad \begin{cases} \displaystyle\int\limits_{\Omega} D(\boldsymbol{w}) : D(\boldsymbol{v}) \; = \; \int\limits_{\Omega} \boldsymbol{F} : D(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in [W^{1,\,q}_{0,\sigma}(\Omega)]^n, \\ \|D(\boldsymbol{w})\|_{L^2} \; \leq \; \|\boldsymbol{F}\|_{L^2}, \\ \|D(\boldsymbol{w})\|_{L^q} \; \leq \; c(q)\|\boldsymbol{F}\|_{L^q} \quad (2 < q < 3 + \varepsilon). \end{cases}$$

---

[5] $[L^q_{\mathrm{sym}}(\Omega)]^{n^2} = \Big\{\boldsymbol{F} = \{F_{ij}\} \in [L^q(\Omega)]^{n^2}; F_{ij} = F_{ji} \text{ a. e. in } \Omega \ (i,j = 1,\dots,n)\Big\}.$

Before turning to the proof we note that for $\boldsymbol{u} \in [W^{1,q}_{0,\sigma}(\Omega)]^n$ and $\boldsymbol{v} \in [W^{1,q'}_{0,\sigma}(\Omega)]^n$ there holds

$$\int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \;=\; 2 \int_\Omega D(\boldsymbol{u}) : D(\boldsymbol{v}).$$

This is readily seen for smooth $\boldsymbol{u}$ and $\boldsymbol{v}$, and then extended to the case of $\boldsymbol{u}$ and $\boldsymbol{v}$ under consideration by observing that

$$\{\boldsymbol{\varphi} \in [C^\infty_c(\Omega)]^n; \operatorname{div} \boldsymbol{\varphi} \;=\; 0 \text{ in } \Omega\}$$

is dense in $[W^{1,q}_{0,\sigma}(\Omega)]^n$ (see Sohr [12; Lemma 2.2.3, pp. 77]).
Next, for symmetric matrices $\boldsymbol{a} = \{a_{ij}\}$ and $\boldsymbol{\varphi} \in [W^{1,q}_{0,\sigma}(\Omega)]^n$ we have

$$\boldsymbol{a} : \nabla \boldsymbol{\varphi} \;=\; \boldsymbol{a} : D(\boldsymbol{\varphi}) \qquad \text{a. e. in} \quad \Omega.$$

Proof of Proposition 1′ If $q = 2$, the statement of Prop. 1′ follows by the same argument as in 3.1 above. If $2 < q < 3 + \delta$, the statement is easily obtained by combining Korn's inequality and Prop. 1 with $2\boldsymbol{F}$ in place of $\boldsymbol{F}$. ■

$\boxed{3.3}$ Let q be as in Prop. 1′. For $\boldsymbol{F} \in [L^q_{\text{sym}}(\Omega)]^{n^2}$ we define

$$\mathcal{S} : \boldsymbol{F} \to \mathcal{S}(\boldsymbol{F}) := D(\boldsymbol{w})$$

where $\boldsymbol{w} \in [W^{1,q}_{0,\sigma}(\Omega)]^n$ is uniquely determined by Prop. 1′. Thus, $\mathcal{S}$ is a linear mapping of $[L^q_{\text{sym}}(\Omega)]^{n^2}$ into itself which satisfies

$$\left\{ \begin{aligned} &\|\mathcal{S}(\boldsymbol{F})\|_{L^2} \;\le\; \|\boldsymbol{F}\|_{L^2}, \\ &\|\mathcal{S}(\boldsymbol{F})\|_{L^q} \;\le\; c(q)\|\boldsymbol{F}\|_{L^q} \quad (2 < q < 3 + \varepsilon). \end{aligned} \right. \tag{3.12}$$

Let $2 < r < q$. From (3.12) and the Riesz-Thorin interpolation theorem it follows that

$$\|\mathcal{S}(\boldsymbol{F})\|_{L^r} \;\le\; [c(q)]^{q(r-2)/r(q-2)} \|\boldsymbol{F}\|_{L^r} \quad \forall\, \boldsymbol{F} \in [L^r_{\text{sym}}(\Omega)]^{n^2}. \tag{3.13}$$

In other words, given $\boldsymbol{F} \in [L^r_{\text{sym}}(\Omega)]^{n^2}$, there exists exactly one $\boldsymbol{w} \in [W^{1,r}_{0,\sigma}(\Omega)]^n$ such that

$$\left\{ \begin{aligned} &\int_\Omega D(\boldsymbol{w}) : D(\boldsymbol{v}) \;=\; \int_\Omega \boldsymbol{F} : D(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in [W^{1,r'}_{0,\sigma}(\Omega)]^n, \\ &\|D(\boldsymbol{w})\|_{L^r} \;\le\; [c(q)]^{q(r-2)/r(q-2)} \|\boldsymbol{F}\|_{L^r}. \end{aligned} \right. \tag{3.14}$$

■

4. A Meyers' type estimate for weak solutions to a generalized stationary Stokes system

Proposition 2 *Let $\partial\Omega$ be Lipschitz. Let $2 < q < 3 + \varepsilon$ ($\varepsilon > 0$ according to Brown; Shen [2]). Fix $r \in]2, q[$ such that*

$$[c(q)]^{q(r-2)/r(q-2)}\Big(1 - \frac{\alpha_1}{\alpha_2}\Big) < 1.$$

Define

$$C_0 = C_0(q,r) := [c(q)]^{q(r-2)/r(q-2)}.$$

Then, for every $\boldsymbol{G} \in [L^r(\Omega)]^{n^2}$ there exists exactly one $\boldsymbol{z} \in [W^{1,r}_{0,\sigma}(\Omega)]^n$ such that

$$\tag{4.1} \begin{cases} \displaystyle\int_\Omega \boldsymbol{A}(\boldsymbol{z}, \boldsymbol{v}) \; = \; \int_\Omega \boldsymbol{G} : \nabla \boldsymbol{v} \quad \forall\, \boldsymbol{v} \in [W^{1,r'}_{0,\sigma}(\Omega)]^n, \\ \Big(1 - C_0\Big(1 - \dfrac{\alpha_1}{\alpha_2}\Big)\Big)\|D(\boldsymbol{z})\|_{L^r} \;\le\; C_1 \|\boldsymbol{G}\|_{L^r}, \end{cases}$$

where $C_1 = C_1(q,r) =$ const.

This result has been presented in Kaplický; Málek; Stará [7] for domains Ω of class $\mathcal{C}^{1,1}$.

To begin with, in (4.1) we pass from $\boldsymbol{G} \in [L^r(\Omega)]^{n^2}$ to $\boldsymbol{G}^* \in [L^r_{\mathrm{sym}}(\Omega)]^{n^2}$ as follows. By Prop. 1 there exists exactly one $\boldsymbol{w}^* \in [W^{1,r}_{0,\sigma}(\Omega)]^n$ such that

$$\begin{aligned} \int_\Omega \nabla \boldsymbol{w}^* : \nabla \boldsymbol{v} \; &= \; \int_\Omega \boldsymbol{G} : \nabla \boldsymbol{v} \quad \forall\, \boldsymbol{v} \in [W^{1,r'}_{0,\sigma}(\Omega)]^n, \\ \|\nabla \boldsymbol{w}^*\|_{L^r} \; &\le \; c(r)\|\boldsymbol{G}\|_{L^r}. \end{aligned}$$

Define $\boldsymbol{G}^* := 2D(\boldsymbol{w}^*)$. It follows

$$\tag{4.2} \begin{cases} \boldsymbol{G}^* \in [L^r_{\mathrm{sym}}(\Omega)]^{n^2}, \\ \displaystyle\int_\Omega \boldsymbol{G} : \nabla \boldsymbol{v} \; = \; \int_\Omega \boldsymbol{G}^* : D(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in [W^{1,r'}_{0,\sigma}(\Omega)]^n, \\ \|\boldsymbol{G}^*\|_{L^r} \;\le\; c^*(r)\|\boldsymbol{G}\|_{L^r} \quad \text{[by Korn's inequality]}. \end{cases}$$

Proof of Proposition 2 (see Clain; Touzani [3 ; Appendix], Giaquinta [6 ; pp. 154-157], Meyers [9]).

1. Existence Define

$$\tilde{H}^{kl}_{ij} := \delta_{ik}\delta_{jl} - \frac{1}{\alpha_2} A^{kl}_{ij} \quad (i,j,k,l = 1, \ldots, n).$$

Then

$$(\widetilde{\boldsymbol{H}}\boldsymbol{\xi})_{ij} \;=\; \tilde{H}^{kl}_{ij}\xi_{kl} = \xi_{ij} - \frac{1}{\alpha_2}A^{kl}_{ij}\xi_{kl}, \quad \boldsymbol{\xi} \in \mathbb{R}^{n^2}.$$

Thus, by the first symmetry property of the coefficients (see (1.4_2)), $\{\boldsymbol{\xi},\boldsymbol{\eta}\} \mapsto (\widetilde{\boldsymbol{H}}\boldsymbol{\xi}) : \boldsymbol{\eta}$ is a symmetric bilinear form on $\mathbb{R}^{n^2} \times \mathbb{R}^{n^2}$. Observing (1.4_3) we find

$$0 \;\le\; (\widetilde{\boldsymbol{H}}\boldsymbol{\xi}) : \boldsymbol{\xi} \;=\; |\boldsymbol{\xi}|^2 - \frac{1}{\alpha_2}A^{kl}_{ij}\xi_{kl}\xi_{ij} \;\le\; \Big(1 - \frac{\alpha_1}{\alpha_2}\Big)|\boldsymbol{\xi}|^2,$$

$$|\widetilde{\boldsymbol{H}}\boldsymbol{\xi}| \;\le\; \Big(1 - \frac{\alpha_1}{\alpha_2}\Big)|\boldsymbol{\xi}|$$

for all $\boldsymbol{\xi} \in \mathbb{R}^{n^2}$.

Now, for $\boldsymbol{w} \in [W^{1,r}_{0,\sigma}(\Omega)]^n$ define

$$\begin{aligned} H_{ij} \;&=\; H_{ij}(\boldsymbol{w}) := (\widetilde{\boldsymbol{H}}(D(\boldsymbol{w})))_{ij} \;+\; \frac{1}{\alpha_2}G^*_{ij} \\ &=\; D_{ij}(\boldsymbol{w}) - \frac{1}{\alpha_2}A^{kl}_{ij}D_{kl}(\boldsymbol{w}) \;+\; \frac{1}{\alpha_2}G^*_{ij} \end{aligned}$$

a. e. in $\Omega\,(i,j = 1,\dots,n)$. Here $\boldsymbol{G}^* = \{G^*_{ij}\}$ is determined by (4.2). The second symmetry property of the coefficients A^{kl}_{ij} (see (1.4_2)) implies $A^{kl}_{ij}D_{kl}(\boldsymbol{w}) = A^{kl}_{ji}D_{kl}(\boldsymbol{w})$ a. e. in Ω. Thus, $\boldsymbol{H} = \{H_{ij}\} \in [L^r_{\text{sym}}(\Omega)]^{n^2}$.

From (3.14) [with $\boldsymbol{F} = \boldsymbol{H}$] it follows that there exists exactly one $\boldsymbol{z} \in [W^{1,r}_{0,\sigma}(\Omega)]^n$ such that

$$\int\limits_\Omega D(\boldsymbol{z}) : D(\boldsymbol{v}) \;=\; \int\limits_\Omega \boldsymbol{H} : D(\boldsymbol{v}) \;=$$

$$= \int\limits_\Omega \Big(D(\boldsymbol{w}) : D(\boldsymbol{v}) - \frac{1}{\alpha_2}\boldsymbol{A}(\boldsymbol{w},\boldsymbol{v}) + \frac{1}{\alpha_2}\boldsymbol{G}^* : D(\boldsymbol{v})\Big) \quad \forall \in [W^{1,r'}_{0,\sigma}(\Omega)]^n, \tag{4.3}$$

$$\|D(\boldsymbol{z})\|_{L^r} \;\le\; C_0\|\boldsymbol{H}\|_{L^r} \;\le\; C_0\bigg(\Big(1 - \frac{\alpha_1}{\alpha_2}\Big)\|D(\boldsymbol{w})\|_{L^r} \;+\; \frac{1}{\alpha_2}\|\boldsymbol{G}^*\|_{L^r}\bigg). \tag{4.4}$$

To proceed, define $\mathcal{T} : \boldsymbol{w} \mapsto \mathcal{T}(\boldsymbol{w}) = \boldsymbol{z}$. The mapping is a contraction of the Banach space $[W^{1,r}_{0,\sigma}(\Omega)]^n$ into itself. Indeed, let $\boldsymbol{w}_i \in [W^{1,r}_{0,\sigma}(\Omega)]^n$, and $\boldsymbol{z}_i = \mathcal{T}(\boldsymbol{w}_i)\,(i = 1,2)$. We obtain

$$\int\limits_\Omega D(\boldsymbol{z}_1 - \boldsymbol{z}_2) : D(\boldsymbol{v}) \;=$$

$$= \int\limits_\Omega \Big(D(\boldsymbol{w}_1 - \boldsymbol{w}_2) : D(\boldsymbol{v}) - \frac{1}{\alpha_2}\boldsymbol{A}(\boldsymbol{w}_1 - \boldsymbol{w}_2,\boldsymbol{v})\Big) \quad \forall\,\boldsymbol{v} \in [W^{1,r'}_{0,\sigma}(\Omega)]^n.$$

It follows

$$
\begin{aligned}
\|D(\mathcal{T}(\boldsymbol{w}_1)-\mathcal{T}(\boldsymbol{w}_2))\|_{L^r} &= \|D(\boldsymbol{z}_1-\boldsymbol{z}_2)\|_{L^r}\\
&\le C_0\Big(1-\frac{\alpha_1}{\alpha_2}\Big)\|D(\boldsymbol{w}_1-\boldsymbol{w}_2)\|_{L^r},
\end{aligned}
$$

where $C_0\Big(1-\dfrac{\alpha_1}{\alpha_2}\Big)<1$ by virtue of our choice of $r\in]2,q[$.

Thus, there exists exactly one $\boldsymbol{z}\in[W^{1,r}_{0,\sigma}(\Omega)]^n$ such that $\mathcal{T}(\boldsymbol{z})=\boldsymbol{z}$. By (4.2) and (4.3) and (4.4),

$$
\int\limits_\Omega \boldsymbol{A}(\boldsymbol{z},\boldsymbol{v}) \;=\; \int\limits_\Omega \boldsymbol{G}^*:D(\boldsymbol{v}) \;=\; \int\limits_\Omega \boldsymbol{G}:\nabla\boldsymbol{v} \quad \forall\,\boldsymbol{v}\in[W^{1,r'}_{0,\sigma}(\Omega)]^n
$$

resp.

$$
\Big(1-C_0\Big(1-\frac{\alpha_1}{\alpha_2}\Big)\Big)\|D(\boldsymbol{z})\|_{L^r} \;\le\; \frac{C_0}{\alpha_2}\|\boldsymbol{G}^*\|_{L^r} \;\le\; C_1\|\boldsymbol{G}\|_{L^r},
$$

where $C_1:=\dfrac{1}{\alpha_2}c^*(r)C_0$.

2. Uniqueness Assume $\widehat{\boldsymbol{z}}\in[W^{1,r}_{0,\sigma}(\Omega)]^n$ satisfies

$$
\int\limits_\Omega \boldsymbol{A}(\widehat{\boldsymbol{z}},\boldsymbol{v}) \;=\; 0 \quad \forall\,\boldsymbol{v}\in[W^{1,r'}_{0,\sigma}(\Omega)]^n.
$$

We may insert $\boldsymbol{v}=\widehat{\boldsymbol{z}}$ (for $r>2$). Hence $\widehat{\boldsymbol{z}}=0$ a.e. in Ω.

■

5. Proof of Theorem 1

Let $\boldsymbol{u}\in[W^{1,2}_{0,\sigma}(\Omega)]^n$ be any weak solution to (1.1)-(1.3). Inserting $\boldsymbol{v}=\boldsymbol{u}$ into (1.6) gives

$$
\alpha_1\int\limits_\Omega |D(\boldsymbol{u})|^2 \;\le\; \int\limits_\Omega A(\boldsymbol{u},\boldsymbol{u}) \;=\; \int\limits_\Omega \boldsymbol{f}:\nabla\boldsymbol{u}.
$$

Hence

$$
\|D(\boldsymbol{u})\|_{L^2} \;\le\; c\|\boldsymbol{f}\|_{L^2}. \tag{5.1}
$$

For what follows, we rewrite integral identity (1.6) in the form

$$
\int\limits_\Omega \boldsymbol{A}(\boldsymbol{u},\boldsymbol{v}) \;=\; \int\limits_\Omega (\boldsymbol{f}+(\boldsymbol{u}\otimes\boldsymbol{u})):\nabla\boldsymbol{v} \quad \forall\,\boldsymbol{v}\in[W^{1,2}_{0,\sigma}(\Omega)]^n
$$

where

$$
\boldsymbol{u}\otimes\boldsymbol{u} \;=\; \{u_iu_j\}.
$$

Define

$$s := \begin{cases} q & \text{if} \quad n = 2, \\ \min\{q, 3\} & \text{if} \quad n = 3; \end{cases}$$

$$\boldsymbol{G} := \boldsymbol{f} \; + \; (\boldsymbol{u} \otimes \boldsymbol{u}) \quad \text{a. e. in} \quad \Omega.$$

By Sobolev's embedding theorem,

$$\|\boldsymbol{u} \otimes \boldsymbol{u}\|_{L^s} \; \le \; c \|D(\boldsymbol{u})\|_{L^2}^2.$$

Fix $r \in]2, s[$ such that

$$[c(s)]^{s(r-2)/r(s-2)} \Big(1 - \frac{\alpha_1}{\alpha_2}\Big) < 1.$$

We now apply Prop. 2 with s in place of q, and $\boldsymbol{G} = \boldsymbol{f} + (\boldsymbol{u} \otimes \boldsymbol{u})$. We obtain the existence and uniqueness of a $\boldsymbol{z} \in [W^{1,r}_{0,\sigma}(\Omega)]^n$ such that

$$\int_\Omega \boldsymbol{A}(\boldsymbol{z}, \boldsymbol{v}) \; = \; \int_\Omega \boldsymbol{G} : \nabla \boldsymbol{v} \quad \forall \, \boldsymbol{v} \in [W^{1,r'}_{0,\sigma}(\Omega)]^n \tag{5.2}$$

$$\Big(1 - C_0\Big(1 - \frac{\alpha_1}{\alpha_2}\Big)\Big) \|D(\boldsymbol{z})\|_{L^r} \; \le \; C_1 \|\boldsymbol{G}\|_{L^r} \; \le \; C_2 (\|\boldsymbol{f}\|_{L^r} \; + \; \|\boldsymbol{f}\|_{L^2}^2) \quad \text{[by (5.1)]}, \tag{5.3}$$

where

$$C_0 = C_0(s, r) := [c(s)]^{s(r-2)/r(s-2)}.$$

Finally, subtracting (1.6′) and (5.2) gives

$$\int_\Omega \boldsymbol{A}(\boldsymbol{u} - \boldsymbol{z}, \boldsymbol{v}) \; = \; 0 \quad \forall \, \boldsymbol{v} \in [W^{1,2}_{0,\sigma}(\Omega)]^n.$$

Whence $\boldsymbol{u} = \boldsymbol{z}$ a. e. in Ω.

■

6. Higher integrability of $D(\boldsymbol{u})$ for small $1 - \dfrac{\alpha_1}{\alpha_2}$

Theorem 2 *Assume*

$$\partial\Omega \quad \textit{is Lipschitz},$$
$$2 < q < 3 + \varepsilon \quad (\varepsilon > 0 \textit{ according to Brown; Shen [2]}).$$

Let $c(q)$ *denote the constant in* (3.11) *(see Prop.* 1′*). Suppose that*

$$c(q)\Big(1 - \frac{\alpha_1}{\alpha_2}\Big) < 1$$

Let $\boldsymbol{f} \in [L^q(\Omega)]^{n^2}$*. Then, for every weak solution* $u \in [W^{1,2}_{0,\sigma}(\Omega)]^n$ *to* (1.1)-(1.3) *there holds*

$$D(\boldsymbol{u}) \in [L^q(\Omega)]^{n^2},$$

$$\Big(1 - c(q)\Big(1 - \frac{\alpha_1}{\alpha_2}\Big)\Big)\|D(\boldsymbol{u})\|_{L^q} \le \begin{cases} C_1(\|\boldsymbol{f}\|_{L^q} + \|\boldsymbol{f}\|^2_{L^2}) & \text{if} \quad n = 2, \\ C_2(\|\boldsymbol{f}\|_{L^q} + \|\boldsymbol{f}\|_{L^3}\|\boldsymbol{f}\|_{L^2} + \|\boldsymbol{f}\|^3_{L^2}) & \\ & \text{if} \quad n = 3. \end{cases}$$

Proof Let $\boldsymbol{u} \in [W^{1,2}_{0,\sigma}(\Omega)]^n$ be any weak solution to (1.1)-(1.3). Define $\boldsymbol{G} := \boldsymbol{f} + (\boldsymbol{u} \otimes \boldsymbol{u})$ a.e. in Ω.

1^{st} *case:* $n = 2$. Let $2 < q < 3 + \varepsilon$. Clearly, $\boldsymbol{G} \in [L^q(\Omega)]^4$. Let $\boldsymbol{G}^* \in [L^q_{\text{sym}}(\Omega)]^4$ satisfy

$$\int_\Omega \boldsymbol{G} : \nabla \boldsymbol{v} = \int_\Omega \boldsymbol{G}^* : D(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in [W^{1,q'}_{0,\sigma}(\Omega)]^2,$$
$$\|\boldsymbol{G}^*\|_{L^q} \le c^*(q)\|\boldsymbol{G}\|_{L^q} \le c(\|\boldsymbol{f}\|_{L^q} + \|\boldsymbol{f}\|^2_{L^2})$$

(cf. (4.2)).

Let $\boldsymbol{w} \in [W^{1,q}_{0,\sigma}(\Omega)]^2$. By Prop. 1′, there exists exactly one $\boldsymbol{z} \in [W^{1,q}_{0,\sigma}(\Omega)]^2$ such that

$$\int_\Omega D(\boldsymbol{z}) : D(\boldsymbol{v}) =$$
$$= \int_\Omega \Big(D(\boldsymbol{w}) : D(\boldsymbol{v}) - \frac{1}{\alpha_2}\boldsymbol{A}(\boldsymbol{w}, \boldsymbol{v}) + \frac{1}{\alpha_2}\boldsymbol{G}^* : D(\boldsymbol{v})\Big) \quad \forall\, \boldsymbol{v} \in [W^{1,q'}_{0,\sigma}(\Omega)]^2$$

$$\|D(\boldsymbol{z})\|_{L^q} \le c(q)\Big(1 - \frac{\alpha_1}{\alpha_2}\Big)\|D(\boldsymbol{w})\|_{L^q} + \frac{1}{\alpha_2}\|\boldsymbol{G}^*\|_{L^q}$$

(cf. the proof of Prop. 2). It follows that the mapping $\boldsymbol{w} \mapsto \boldsymbol{z}$ possesses a fixed point, $\boldsymbol{z}$ say. Hence

$$\int_\Omega \boldsymbol{A}(\boldsymbol{z}, \boldsymbol{v}) = \int_\Omega \boldsymbol{G} : \nabla \boldsymbol{v} \quad \forall\, \boldsymbol{v} \in [W^{1,q'}_{0,\sigma}(\Omega)]^2.$$

As above (cf. the proof of Th. 1), $\boldsymbol{z} = \boldsymbol{u}$ a.e. in Ω. Whence the claim.

2^{nd} *case:* $n = 3$. Firstly, assume $2 < q \le 3$. Since $\boldsymbol{u} \otimes \boldsymbol{u} \in [L^3(\Omega)]^9$, the assertion of Theorem 2 can be proved by the same arguments as in the case $n = 2$.

Secondly, assume $3 < q < 3 + \varepsilon$. For every $\boldsymbol{F} \in [L^3_{\text{sym}}(\Omega)]^9$ there exists exactly one $\boldsymbol{w} \in [W^{1,3}_{0,\sigma}(\Omega)]^3$ such that

$$\begin{cases} \displaystyle\int_\Omega D(\boldsymbol{w}) : \nabla \boldsymbol{v} = \int_\Omega \boldsymbol{F} : D(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in [W^{1,3/2}_{0,\sigma}(\Omega)]^3, \\ \|D(\boldsymbol{w})\|_{L^3} \le [c(q)]^{q/3(q-2)}\|\boldsymbol{F}\|_{L^3} \end{cases}$$

(cf. (3.14); $r = 3$ therein). Clearly, $\dfrac{q}{3(q-2)} < 1$. Therefore $[c(q)]^{q/3(q-2)} \le c(q)$ [6]. Now Prop. 2 implies

$$\begin{cases} D(\boldsymbol{u}) \in [L^3(\Omega)]^9, \\ \left(1 - [c(q)]^{q/3(q-2)}\left(1 - \dfrac{\alpha_1}{\alpha_2}\right)\right)\|D(\boldsymbol{u})\|_{L^3} \le c(\|\boldsymbol{f}\|_{L^3} + \|\boldsymbol{f}\|^2_{L^2}). \end{cases}$$

It follows $(\boldsymbol{u} \otimes \boldsymbol{u}) \in [L^s(\Omega)]^9$ for all $1 \le s < +\infty$.

Now, fix q_1 such that $3 < q < q_1 < 3+\varepsilon$. Since $\dfrac{q_1(q-2)}{q(q_1-2)} < 1$ we have $[c(q)]^{q_1(q-2)/q(q_1-2)} \le c(q)$. Therefore, again appealing to Prop. 2 we obtain

$$\begin{cases} D(\boldsymbol{u}) \in [L^q(\Omega)]^9, \\ \left(1 - [c(q)]^{q_1(q-2)/q(q_1-2)}\left(1 - \dfrac{\alpha_1}{\alpha_2}\right)\right)\|D(\boldsymbol{u})\|_{L^q} \le c\|f + (\boldsymbol{u} \otimes \boldsymbol{u})\|_{L^q}. \end{cases}$$

Finally, by Sobolev's embedding theorem,

$$\|\boldsymbol{u} \otimes \boldsymbol{u}\|_{L^q} \le c\|D(\boldsymbol{u})\|_{L^3}\|D(\boldsymbol{u})\|_{L^2} \le c(\|\boldsymbol{f}\|_{L^3} + \|\boldsymbol{f}\|^2_{L^2})\|\boldsymbol{f}\|_{L^2}.$$

Whence the claim.

■

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